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COMBINATORIAL DIMENSION AND RANDOM SETS

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ABSTRACT

Having defined the combinatorial dimension of an arbitrary subset of a finite dimensional lattice, for every $\alpha \in (1,2)$ we produce a set in $\mathbf{N}^2$ whose dimension equals α.

I. Introduction

The notion of fractional Cartesian products in a context of harmonic analysis, studied in [1], subsequently gave rise to a notion of combinatorial dimension, further studied in [2]. We recall some definitions from [2]: As usual, $\mathbf{N}$ will denote the set of natural numbers. Let J be a positive integer, $E \subset \mathbf{N}^J$ (= the usual J-fold Cartesian product of $\mathbf{N}$), and define for any positive integer s

$$\Psi_E(s) = \max\{|E \cap (A_1 \times \cdots \times A_J)| : A_i \subset \mathbf{N}, |A_1| = \cdots = |A_J| = s\}.$$

($|\cdot|$ denotes cardinality.) The combinatorial dimension of E is given by

$$\dim E = \inf\left\{a : \overline{\lim_{s\to\infty}} \frac{\bar{\Psi}_E(s)}{s^a} < \infty\right\}.$$

Suppose $\dim E = \alpha$. We say that E is α-dimensional and distinguish between two mutually exclusive cases: E is exactly α-dimensional when

$$\overline{\lim} \frac{\bar{\Psi}_E(s)}{s^\alpha} < \infty,$$

and asymptotically α-dimensional, otherwise. The following is a more stringent notion: $E \subset \mathbf{N}^J$ is an α-Cartesian product, $1 \leqq \alpha \leqq J$, if

$$0 < \underline{\lim_{s}} \left(\frac{\bar{\Psi}_E(s)}{s^\alpha}\right) \leqq \overline{\lim_{s}} \left(\frac{\bar{\Psi}_E(s)}{s^\alpha}\right) < \infty.$$

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Clearly, every α-Cartesian product is exactly α-dimensional. The converse, however, is false (see Proposition C below).

In [1], given arbitrary integers $p \geqq q > 1$, p/q-Cartesian products were displayed as subsets of $\mathbf{N}^{\binom{p}{q}}$; for an arbitrary $1 < \alpha < \infty$, α-dimensional sets were obtained as appropriate 'limits' of rational r-Cartesian products in the 'infinite dimensional' framework $\bigcup_{j=1}^{\infty} \mathbf{N}^j$. A basic question was then raised: Given $1 < \alpha < 2$, can we find α-dimensional sets in $\mathbf{N}^2$? This question is affirmatively answered in this paper.

We thank Professor J. Schmerl for numerous stimulating conversations on topics treated in this paper.

II. Combinatorial dimension of random sets

THEOREM A. (a) *There exists a family of sets* $\{F_x : F_x \subset \mathbf{N}^2, x \in (1,2]\}$ *with the following properties: For each* $x \in (1,2]$

(i) F_x *is exactly* x*-dimensional,*

and

(ii) $F_x = \bigcup_{t<x} F_t$.

(b) *There exists a family of sets* $\{F_x^{\cdot} : F_x \subset \mathbf{N}^2,\ x \in [1,2)\}$ *with the following properties: For each* $x \in [1,2)$

(i) F_x *is asymptotically* x*-dimensional,*

and

(ii) $F_x = \bigcap_{t>x} F_t$.

We prove first a simpler yet archtypical result whose proof contains the main idea for the proof of Theorem A.

PROPOSITION B. *Exactly* α*-dimensional sets exist in* $\mathbf{N}^2$ *for every* $\alpha \in (1,2)$.

Lemma 1 below is a finite version of Proposition B. Denote

$$I(s) = \{A \subset \mathbf{N} : |A| = s\},$$

and

$$J_k = \{1, \cdots, k\}.$$

LEMMA 1. *Let* $1 < \alpha < 2$, *and* M *be an arbitrary positive integer. There exists an integer depending on* M, $n(M) = n \geqq M$ *and* $F \subset J_n \times J_n$ *so that*

$$\Psi_F(n) \geqq \tfrac{1}{2} n^{\alpha}, \tag{1}$$

and

$$\Psi_F(s) \leqq s^\alpha \qquad \textit{for all } s \geqq L(\alpha), \tag{2}$$

where

$$L(\alpha) = \min\{s : 2s - s^\alpha(2-\alpha) \leqq -1\}.$$

PROOF. Let $k \geqq M$ be an arbitrary integer, and $(X_{ij}^{(k)})_{i,j\in\mathbf{N}}$ be an array of independent Bernoulli variables (on $(\Omega, \mathbf{P})$) where

$$\mathbf{P}(X_{ij}^{(k)} = 1) = k^{\alpha-2} \quad \text{and} \quad \mathbf{P}(X_{ij}^{(k)} = 0) = 1 - k^{\alpha-2}.$$

Suppose $L(\alpha) \leqq s \leqq M$, and $A, B \in I(s)$. Clearly,

$$\mathbf{P}\left(\sum_{\substack{i\in A\\ j\in B}} X_{ij}^{(k)} \geqq s^\alpha\right) \leqq \sum_{m=s^\alpha}^{s^2} \binom{s^2}{m} k^{(\alpha-2)m}(1-k^{\alpha-2})^{s^2-m} \leqq k^{(\alpha-2)s^\alpha} 2^{s^2}. \tag{3}$$

Summing over all $A, B \subset J_k$, $A, B \in I(s)$, we deduce from (3) and the definition of $L(\alpha)$

$$\mathbf{P}\left(\sum_{\substack{i\in A\\ j\in B}} X_{ij}^{(k)} \geqq s \text{ for some } A, B \subset J_k,\ A, B \in I(s)\right) \leqq k^{2s} k^{(\alpha-2)s^\alpha} 2^{s^2} \leqq 2^{M^2}/k. \tag{4}$$

By Chebyshev's inequality, we have

$$\mathbf{P}\left(\left|\sum_{i,j\in J_k} X_{ij}^{(k)} - k^\alpha\right| \geqq k\right) \leqq 2k^{\alpha-2}. \tag{5}$$

Combining (4) and (5) for sufficiently large and hereafter fixed k, we obtain with high probability $\omega \in \Omega$ so that for all $L(\alpha) \leqq s \leqq M$

$$\begin{gathered} \sum_{i,j\in J_k} X_{ij}^{(k)}(\omega) \geqq \tfrac{1}{2}k^\alpha \quad \text{and} \quad \sum_{\substack{i\in A\\ j\in B}} X_{ij}^{(k)}(\omega) \leqq s^\alpha \\ \text{for all } A, B \subset J_k, \quad A, B \in I(s). \end{gathered} \tag{$*$}$$

We thus obtain a random set

$$F_\omega = \{(i,j) \in J_k \times J_k : X_{ij}^{(k)}(\omega) = 1\},$$

which, by $(*)$, satisfies

$$\Psi_{F_\omega}(k) \geqq \tfrac{1}{2}k^\alpha, \tag{6}$$

and

$$\Psi_{F_\omega}(s) \leqq s^\alpha \qquad \text{for all } L(\alpha) \leqq s \leqq M. \tag{7}$$

Let

$$n = \min\{j \geqq M : \Psi_{F_\omega}(j) \geqq \tfrac{1}{2}j^\alpha\}$$

(from (6): $M \leqq n \leqq k$); by a judicious removal of $[\Psi_{F_\omega}(n) - \frac{1}{2}n^\alpha - 1]$ points from $F \subset J_k \times J_k$, and by relabeling coordinates we obtain $F \subset J_n \times J_n$ that satisfies (1) and (2). □

The following lemma is an elementary fact that we formalize for later use. $F_1, F_2 \subset \mathbf{N}^2$ are said to be bidisjoint if $\pi_1(F_1) \cap \pi_1(F_2) = \pi_2(F_1) \cap \pi_2(F_2) = \varnothing$, where π_1 and π_2 are canonical projections from $\mathbf{N}^2$ onto $\mathbf{N}$.

LEMMA 2. *Let $\{F_j\}_{j=1}^\infty$ be a collection of mutually bidisjoint sets so that for each j*

$$\Psi_{F_j}(s) \leqq Ks^\alpha \qquad \textit{for all } s \geqq 1.$$

Let

$$F = \bigcup_{j=1}^\infty F_j.$$

Then,

$$\Psi_F(s) \leqq 2Ks^\alpha \qquad \textit{for all } s \geqq 1.$$

PROOF. Suppose that $F_j \subset I_j^{(1)} \times I_j^{(2)}$, where $I_j^{(l)} \cap I_k^{(l)} = \varnothing$ whenever $j \neq k$, $l = 1,2$. Let $A, B \in I(s)$, and write $A_j = A \cap I_j^{(1)}$, $B_j = B \cap I_j^{(2)}$. Then,

$$\begin{aligned} |F \cap (A \times B)| &= \sum_j |F \cap (A_j \times B_j)| \\ &\leqq K \sum_j (\max(|A_j|, |B_j|))^\alpha \\ &\leqq K\left\{\left(\sum_j |A_j|\right)^\alpha + \left(\sum_j |B_j|\right)^\alpha\right\} \\ &\leqq 2Ks^\alpha. \end{aligned}$$

□

PROOF OF PROPOSITION B. The assertion is trivial for $\alpha = 1,2$. Having fixed $\alpha \in (1,2)$, for each $j \geqq 1$ let $n(j)$ be as in Lemma 1. Let $\{I_j\}$ be a sequence of mutually disjoint subsets of $\mathbf{N}$, $|I_j| = n(j)$. By Lemma 1, we find $F_j \subset I_j \times I_j$ so that

$$\Psi_{F_j}(s) \leqq Ks^\alpha \qquad \text{for all } s \geqq 1,$$

where

$$K = (L(\alpha))^2,$$

and

$$\Psi_{F_j}(n(j)) \geqq \tfrac{1}{2} n(j)^{\alpha}.$$

By Lemma 2, $F = \bigcup_{j=1}^{\infty} F_j$ is exactly α-dimensional. □

Let Φ be any function increasing to ∞ satisfying the following two properties:

(a) $\lim_{s\to\infty} \Phi(s)/s = \infty$,

and

(b) $\overline{\lim}_{s\to\infty} \Phi(s)/s^{2-\varepsilon} = 0$ for some $\varepsilon > 0$.

Next, let k be an arbitrarily large integer, and $(X_{ij}^{(k)})_{i,j\in\mathbf{N}}$ be an array of independent Bernoulli variables where

$$\mathbf{P}(X_{ij}^{(k)} = 1) = \Phi(k)/k^2.$$

By appropriately modifying the computations that led to Lemma 1, we obtain the following (details are omitted):

PROPOSITION C. *Let Φ be as above. There exist $F \subset \mathbf{N}^2$ and $K_1, K_2 > 0$ with the following properties*:

(i) $\Psi_F(s) \leqq K_1\Phi(s)$

for all integers $s \geqq 1$, and

(ii) $\Psi_F(k_j) \geqq K_2\Phi(k_j)$

for some (k_j), $k_j \to \infty$.

In particular, for every $\alpha \in [1,2)$ there exist asymptotically α-dimensional sets in $\mathbf{N}^2$.

We now proceed to the proof of Theorem A. First, we require a refinement of Lemma 1.

LEMMA 3. *Let $1 < \alpha < \beta < 2$. For each positive integer M there exists an integer (depending on M, α and β) $k(M;\alpha,\beta) = k \geqq M$ so that the following holds: Let $\gamma \in [\alpha,\beta]$ be arbitrary. There exists an integer n, $M \leqq n \leqq k$, and $F \subset J_n \times J_n$ so that*

$$\Psi_F(n) \geqq \tfrac{1}{2} n^{\gamma}, \tag{8}$$

and

$$\Psi_F(s) \leqq s^{\gamma} \qquad \text{for all } s \geqq L(\gamma) \tag{9}$$

($L(\gamma) = \min\{s : 2s - s^{\gamma}(2-\gamma) \leqq -1\}$). *Furthermore, $F \subset J_n \times J_n$ satisfies also the following*:

(10) *Let $\{q_j\}_{j=1}^{\infty}$ be an enumeration of the rationals in $[\alpha, 2)$. Then,*

$$\Psi_F(s) \leqq s^{q_j} \qquad \textit{for all } s \geqq L_\gamma(q_j) \textit{ and } q_j \geqq \gamma,$$

where

$$L_\gamma(q_j) = \min\{s : 2s - s^{q_j}(2-\gamma) \leqq -1\}.$$

PROOF. We argue as in the proof of Lemma 1: Let $(X_{ij}^{(k)})_{i,j\in\mathbf{N}}$ be an array of independent Bernoulli variables on $(\Omega, \mathbf{P})$ where

$$\mathbf{P}(X_{ij}^{(k)} = 1) = k^{\gamma-2}.$$

Next, consider the event

$$H_\gamma^{(k)} = \left\{ \sum_{\substack{i\in A\\ j\in B}} X_{ij}^{(k)} \geqq s^\gamma \text{ for some } A, B \subset J_k,\ L(\gamma) \leqq |A| = |B| = s \leqq M \right\}.$$

Following (4) (in the proof of Lemma 1), we obtain

$$\mathbf{P}(H_\gamma^{(k)}) \leqq M2^{M^2}/k.$$

Similarly, for $q_j \geqq \gamma$ let

$$H_{q_j}^{(k)} = \left\{ \sum_{\substack{i\in A\\ j\in B}} X_{ij}^{(k)} \geqq s^{q_j} \text{ for some } A, B \subset J_k,\ L_\gamma(q_j) \leqq |A| = |B| = s \leqq M \right\},$$

and deduce that

$$\mathbf{P}(H_{q_j}^{(k)}) \leqq M2^{M^2}/k. \tag{11}$$

Now, observe that as a function of q_j, $L_\gamma(\cdot)$ is a decreasing function and hence $\{L_\gamma(q_1), \cdots, L_\gamma(q_j), \cdots\}$ is a finite set that we enumerate as

$$\{L_1, \cdots, L_N\},$$

where $L_1 = L_\gamma(2)$ and $L_N = L(\gamma)$. Let $1 \leqq l \leqq N$ and write

$$Q_l = \{q_j : L_\gamma(q_j) = L_l\}.$$

Let $q_{j_1}, \cdots, q_{j_m} \in Q_l$ be arbitrary, $q_{j_1} > \cdots > q_{j_m} > \gamma$. Clearly,

$$H_{q_{j_1}}^{(k)} \subset \cdots \subset H_{q_{j_m}}^{(k)},$$

and therefore from (11) we obtain

$$\mathbf{P}\left(\bigcap_{q_j \in Q_l} \sim H_{q_j}^{(k)} \right) \geqq 1 - M2^{M^2}/k.$$

On the other hand, as in (5) an application of Chebyshev's inequality yields

(12) $$\mathbf{P}\left(\left|\sum_{i,j\in J_k} X_{ij}^{(k)} - k^{\gamma}\right| \geqq k\right) \leqq 2k^{\gamma-2}.$$

We now choose k large enough so that we can (with high probability) find ω in

$$G_{\gamma,k} = \left[\bigcap_{Q_1} \sim H_{q_j}^{(k)}\right] \cap \cdots \cap \left[\bigcap_{Q_l} \sim H_{q_j}^{(k)}\right] \cap \left\{\sum_{i,j\in J_k} X_{ij}^{(k)} \geqq \tfrac{1}{2}k^{\gamma}\right\}.$$

At this point we claim that $k = k(M;\alpha,\beta)$ can be chosen and fixed so that given any $\gamma \in [\alpha,\beta]$, we can (with high probability) find $\omega \in G_{\gamma,k}$. To see this, we observe that the size of k depends on $N = |\{L_\gamma(q_j)\}_{j=1}^{\infty}|$, on $k^{2-\gamma}$ and on $\min\{k : k^{\gamma} \geqq 2k\}$. Our claim follows from three simple facts: There is N_0 so that $N_0 \geqq |\{L_\gamma(q_j)\}_{j=1}^{\infty}|$ for all $\gamma \in [\alpha,\beta]$; $k^{2-\gamma} \geqq k^{2-\beta}$;

$$\min\{k : k^{\beta} \geqq 2k\} \leqq \min\{k : k^{\gamma} \geqq 2k\}.$$

Having fixed a large enough k that depends only on M, α, and β, we obtain a random set

$$F_\omega = \{(i,j) \in J_k \times J_k : X_{ij}^{(k)}(\omega) = 1\}$$

($\omega \in G_{\gamma,k}$), and conclude the proof as we did in Lemma 1. □

Proof of Theorem A. Part (a). We start with an arbitrary sequence $(\alpha_j)_{-\infty<j<\infty}$: $\alpha_0 = 3/2$, α_j converging to 2 monotonically as $j \to +\infty$, and α_j converging to 1 monotonically as $j \to -\infty$. Let

$$\{R_{ij} : 0 < i < \infty,\ -\infty < j < \infty\}$$

be a partition of $\mathbf{N}$, where $|R_{ij}| = \infty$ for each i and j. For each $-\infty < j < \infty$, let $\{r_{ij}\}_{i=1}^{\infty}$ be an enumeration of the rationals in $[\alpha_j, \alpha_{j+1})$, and for each i let

$$\tilde{F}_{ij} \subset R_{ij} \times R_{ij}$$

be an exactly r_{ij}-dimensional set that is obtained by an application of Lemma 3 in the following way: Let $\{I_m\}_{m=1}^{\infty}$ be a sequence of mutually disjoint sets in R_{ij}, $|I_m| = n(m, r_{ij})$, where

$$m \leqq n(m, r_{ij}) = n_m \leqq k(m; \alpha_j, \alpha_{j+1}),$$

and $F^{(m)} \subset I_m \times I_m$ are obtained by an application of Lemma 3. Let

$$\tilde{F}_{ij} = \bigcup_{m=1} F^{(m)}.$$

By Lemma 3 and Lemma 2, $\tilde{F}_{ij} \subset R_{ij} \times R_{ij}$ satisfies the following:

(13) $\psi_{\tilde{F}_{ij}}(s) \leqq 2K_{ij}(q)s^q$ for all $s \geqq 1$ and rationals $q \geqq r_{ij}$,

where

$$K_{ij}(q) = (L_{r_{ij}}(q))^2$$

and

(14) $$\Psi_{\tilde{F}_{ij}}(n_m) \geqq \tfrac{1}{2} n_m^{r_{ij}},$$

for all m. For each $x \in (1,2]$, define

$$F_x = \bigcup_{r_{ij} < x} \tilde{F}_{ij}.$$

CLAIM 1. *There is $K_x > 0$ so that*

$$\Psi_{F_x}(s) \leqq K_x s^x \qquad \textit{for all } s \geqq 1.$$

Suppose $x \in [\alpha_j, \alpha_{j+1})$. Let $A, B \in I(s)$ and write $A_{ij} = A \cap R_{ij}$, $B_{ij} = B \cap R_{ij}$. Since $\{R_{ij} \times R_{ij}\}$ is a collection of mutually bidisjoint subsets of $\mathbf{N}^2$, we have

$$|F_x \cap (A \times B)| = \sum_{i,j} |\tilde{F}_{ij} \cap (A_{ij} \times B_{ij})|.$$

Let

$$S = \{(i,j) : |\tilde{F}_{ij} \cap (A_{ij} \times B_{ij})| \neq 0\}$$

(clearly $|S| < \infty$), and let $q < x$ be a rational so that

(i) $q > r_{ij}$,

and

(ii) $L_{r_{ij}}(q) = L_{r_{ij}}(x)$

whenever $(i,j) \in S$. It follows from (13) and (15) (see Lemma 3) that

$$|F_x \cap (A \times B)| \leqq 2 \max_{(i,j) \in S} (L_{r_{ij}}(x))^2 s^x.$$

But, $L_{r_{ij}}(x) \leqq L(x)$ for all $r_{ij} \leqq x$, and the claim is established ($K_x = 2(L(x))^2$).

CLAIM 2. $\overline{\lim}_s \Psi_{F_x}(x)/s^x \geqq \frac{1}{2}$.

Let $m \geqq 1$ be arbitrary. Since

$$m \leqq n(m, r_{ij}) \leqq k(m; \alpha_j, \alpha_{j+1})$$

for all i, we obtain an increasing sequence of rationals $(q_i)_{i=1}^{\infty} \subset [\alpha_j, \alpha_{j+1})$ so that

(i) $q_i \to x$ as $i \to \infty$,

and

(ii) $n(m,q_i)=n_0 \geqq m$ for all i.

Therefore, by (14),

$$\Psi_{F_x}(n_0) \geqq \tfrac{1}{2} n_0^{q_i} \qquad \text{for all } i,$$

and the claim follows.

The definition of F_x and the two claims above imply that $\{F_x\}_{x\in(1,2]}$ satisfies the required properties.

The proof of part (b) is similar and is omitted. □

Remarks. (1) The existence of α-dimensional sets within $\mathbf{N}^2$ was obtained in this paper by random methods. J. Schmerl pointed out to us that explicit designs of $\frac{3}{2}$ and $\frac{5}{3}$-Cartesian products in $\mathbf{N}^2$ can be obtained as corollaries to graph theoretic results (theorem 2.8 and theorem 2.9 on p. 314 of [3]) which aim at the open problem of Zarankiewicz (whose statement can be found on p. 309 of [3]). For any other $1<\alpha<2$, explicit designs of α-dimensional sets and afortiori α-Cartesian products in $\mathbf{N}^2$ are not known.

(2) (We assume here that the reader has some familiarity with commutative harmonic analysis.) We say that a spectral set E in a discrete abelian group is p-Sidon, $1 \leqq p \leqq 2$ if

$$C_E(\hat{\Gamma})^\wedge \subset l^r \tag{15}$$

holds precisely when $r \geqq p$. E is said to be asymptotic p-Sidon if (15) above holds precisely when $r > p$. (Notation: $C_E(\hat{\Gamma})$ = space of continuous functions on $\hat{\Gamma}$ whose Fourier transform is supported in E.) Next, we recall that $F=\{\chi_j\}_{j=1}^{\infty} \subset \Gamma$ is dissociate if for any $\{\chi_j\}_{j=1}^{N} \subset F$ and $\delta_j = 0, \pm 1, \pm 2$, the relation

$$\prod_{j=1}^{N} \chi_j^{\delta_j} = 1 \quad (=\text{identity element in } \Gamma)$$

holds precisely when $\delta_j = 0$ for all $j = 1, \cdots, N$.

For every positive integer n, the ordinary n-fold Cartesian products of any dissociate set F are known to be prototypical examples of $2n/(n+1)$-Sidon sets ([4] and [5]). In [1], filling the gaps between $2n/(n+1)$ and $2(n+1)/(n+2)$, appropriate designs of J/K-Cartesian products of F, displayed as subsets of $F^{(J\choose K)}$, turned out to be $2J/(J+K)$-Sidon; for irrational $p \in (1,2)$, p-Sidon and asymptotic p-Sidon sets were obtained as limits of 'rational' Cartesian products in a framework of an 'infinite product' of F. At that point, a natural question remained open: Does F^n contain p-Sidon and asymptotic p-Sidon sets where

$p \in (2(n-1)/n, 2n/(n+1))$ is arbitrary? An affirmative answer is provided via the α-dimensional sets obtained here, and theorem 5.2 of [2] which states essentially the following: $E \subset F^n$ is exactly α-dimensional (asymptotically α-dimensional) iff F is $2\alpha/(\alpha+1)$-Sidon (asymptotic $2\alpha/(\alpha+1)$-Sidon).

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